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TECHNICAL NOTE

On the *k*-3 stagnation point anomaly

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Two equation models predict an anomalously large growth of turbulent kinetic energy in stagnation point flows (Strahle 1985; Launder and Kato 1993). Even when the stagnation point region is not of interest per se, this spurious behavior can upset the rest of the flow computation. The present paper describes one aspect of the stagnation point anomoly, as it occurs eddy-viscosity models based on transport equations for velocity and time scales; it is not meant to be a comprehensive survey of strategies that have been advanced to deal with various other aspects of stagnation point flow.

The usual explanation for the stagnation point anomaly is that the eddy-viscosity formula

$$\overline{u_i u_j} = -2\nu_t S_{ij} + \tfrac{2}{3}k\delta_{ij} \tag{1}$$

gives an erroneous normal stress difference (Launder and Kato 1993). In 1, $S_{ij} = (\partial_i U_j + \partial_j U_i)/2$ is the rate of strain and

$$\nu_t = C_\mu \overline{v^2} T \tag{2}$$

is the eddy viscosity in the form used in two-equation models. $\overline{v^2}$ is the velocity scale — k in the k–ε model — and T is the turbulent time scale — k/ε in the k–ε model, although the Kolmogoroff scale $(\nu/\varepsilon)^{1/2}$ is more suitable near the wall of a high Reynolds number boundary layer (Durbin 1991).

Some of my computations suggest an additional consideration: in these computations, as the stagnation point was approached, T became very large. The ε-equation is of the form

$$\partial_t \varepsilon + U \cdot \nabla\varepsilon = \frac{C_{\varepsilon_1}\mathscr{P} - C_{\varepsilon_2}\varepsilon}{T} + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \nabla\varepsilon \right) \tag{3}$$

where the rate of turbulent energy production is

$$\mathscr{P} = 2\nu_t S_{ij} S_{ji} \tag{4}$$

A large value of T in 3 causes the production of ε to be too small, allowing spuriously high turbulent kinetic energy. The kinetic energy is obtained by solving

$$\partial_t k + U \cdot \nabla k = \mathscr{P} - \varepsilon + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_\kappa} \right) \nabla k \right) \tag{5}$$

The stagnation point anomaly can be ameliorated by imposing the "realizability" constraint $2k \geq \overline{u^2} \geq 0$ upon (1) via a bound on the time scale. The rate of strain tensor S_{ij} is symmetric and becomes purely diagonal in principal-axes coordinates. The diagonal elements, λ_α, $\alpha = 1, \ldots, 3$, are its eigenvalues and satisfy

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = S_{ij} S_{ji} \equiv |\mathbf{S}|^2 \tag{6}$$

In incompressible flow

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \tag{7}$$

It follows from (6) and (7) that

$$|\lambda_\alpha| = \sqrt{|\mathbf{S}|^2/2} \tag{8}$$

in two dimensions (i.e., when $\lambda_3 = 0$), and

$$|\lambda_\alpha| \leq \sqrt{2|\mathbf{S}|^2/3} \tag{9}$$

in three dimensions.

If (1) is written in the principal axes of S_{ij}, it becomes

$$\overline{u_\alpha^2} = -2\nu_t \lambda_\alpha + \tfrac{2}{3}k \tag{10}$$

Of the constraints $\overline{u_\alpha^2} \geq 0$ and $2k \geq \overline{u_\alpha^2}$, $\forall\alpha$ the former is more stringent. [Proof: the former requires

$$2\nu_t \max_\alpha \lambda_\alpha \leq \tfrac{2}{3}k \tag{11}$$

while the latter requires

$$2\nu_t \min_\alpha \lambda_\alpha \geq -\tfrac{4}{3}k$$

(N.B., max $\lambda_\alpha \geq 0$ and min $\lambda_\alpha \leq 0$). From (7)

$$\min \lambda_\alpha \geq -2 \max \lambda_\alpha$$

Multiplying by $2\nu_t$ and assuming that the constraint (11) is satisfied

$$2\nu_t \min \lambda_\alpha \geq -4\nu_t \max \lambda_\alpha$$
$$\geq -\tfrac{4}{3}k$$

Thus, the second constraint follows if the first (i.e., Equation 11) is met.]

Substituting Equation 2 into 11 results in the time-scale bound

$$T \leq \frac{k}{3\overline{v^2}C_\mu} \frac{1}{\max \lambda_\alpha} \tag{12}$$

which gives

$$T \leq \frac{2k}{3\overline{v^2}C_\mu} \frac{1}{\sqrt{2|\mathbf{S}|^2}} \tag{13}$$

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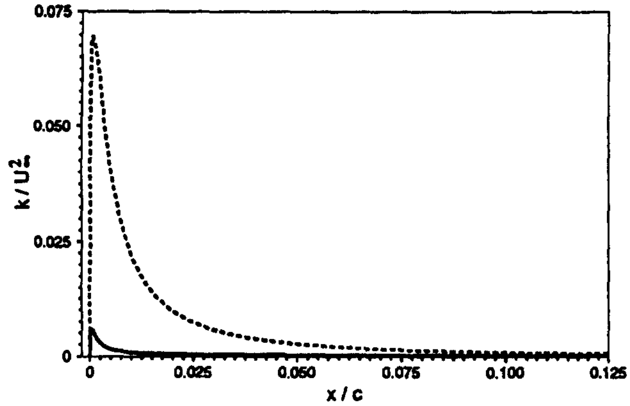

Figure 1 k/U_∞^2 along the stagnation streamline: ———, with 15 imposed; -----, without constraint

in two dimensions, and

$$T \le \frac{2k}{3v^2 C_\mu}\sqrt{\frac{3}{8|\mathbf{S}|^2}} \tag{14}$$

in three dimensions. These bounds might be imposed computationally by using

$$T = \min\left(\frac{k}{\varepsilon}, \frac{2k}{3v^2 C_\mu}\sqrt{\frac{3}{8|\mathbf{S}|^2}}\right) \tag{15}$$

in Equations 2, 3, and elsewhere that T is needed. Any alternative to Equation 15 that is sufficient to ensure 13 and 14 can be used: for instance, the second factor in 15 could be multiplied by an empirical constant < 1. Such a constant could be used to obtain agreement to an experimental datum; but that is beyond the scope of the present brief communication.

By Equations 4 and 11, the rate of energy production becomes linear in the modulus of the mean rate of strain when the bound 12 is attained

$$\mathscr{P} \le \frac{2}{3}k\frac{|\mathbf{S}|^2}{\max \lambda_\alpha}$$

$$= \frac{2}{3}k\sqrt{2|\mathbf{S}|^2} \quad \text{in } 2-D \tag{16}$$

This is significant because one-equation models do not experience a stagnation point anomaly, although they may be motivated by the k–ε model (Baldwin and Barth 1990). The production term in one-equation models is linear in the rate of strain, as is 16.

Launder and Kato (1993) and Menter (1992) avoided the stagnation point anomaly by replacing rate of strain by vorticity in $\mathscr{P}$ so that energy would not grow in a potential flow. This device would cause spurious production in rotating or swirling flow; also, it is erroneous to suppose that turbulence is not amplified by irrotational strains. Menter (1993) imposed a limit on $\mathscr{P}/\varepsilon$, which might correspond to using 16 directly as a constraint. Strahle (1985) noted that if $C_{\varepsilon_1} = C_{\varepsilon_2}$, then a $\mathscr{P} = \varepsilon$ equilibrium could be established in the stagnating flow, thereby avoiding the anomoly. However, in equilibrium straining how $\mathscr{P} > \varepsilon$, so this is not an attractive suggestion per se. All these approaches, as well as the present, can be characterized as preventing uncontrolled growth of $\mathscr{P}/\varepsilon$. Second-moment closure models include evolution equations for the normal stresses, and this permits a more elaborate approach to the stagnation point problem (Craft et al. 1993). The present paper is not addressed to second-moment closure, but 15 might be helpful there, also.

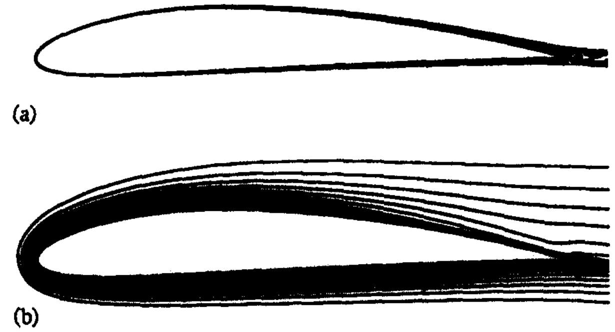

Figure 2 Contours of constant k/U_∞^2; (a), with 15 imposed; (b), without constraint; contour intervals of 1.5×10^{-3}

Figure 1 shows k along the stagnation streamline of a NACA4412 airfoil at zero angle of attack and with $k = 4 \times 10^{-4}$ U_∞^2 in the approach flow. This computation was done with the k–ε–v^2 model (Durbin 1995), which can be integrated to the surface without need for damping functions. The two curves show the solution with and without 15 imposed. The constraint prevents the large growth of κ, although some amplification still occurs. Figure 2 shows contours of constant k for the two cases.

Flat plate, zero, and adverse pressure gradient boundary-layer computations produced exactly the same solution with and without 15 imposed. In the log-layer, the ratio of the second term to the first is $1/\sqrt{3C_\mu} \approx 1.92$, so that $T = k/\varepsilon$ satisfies Equation 12.

Acknowledgments

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